AN APPROXIMATE METHOD OF SOLVING THE PROBLEM OF PLANE VORTEX GAS FLOW

(PRIBLIZHENNYI METOD RESHENIIA ZADACH O PLOSKOM VIKHREVOM DVIZHENII GAZA)

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This article deals with plane steady motion of an ideal compressible fluid. In [1] it has been demonstrated that if a function of the velocity $v(p, \psi)$ (p is the pressure and ψ the stream function) is such that the value of $(1/v)(\partial^2 v/\partial p^2)$ is independent of ψ, the nonlinear equation of vortex motion [2,3] reduces to a linear-potential flow equation; this is a generalisation of the known results due to Rudnev [3]. With adiabatic motion this condition can be approximated if the entropy function $\vartheta(\psi) = p^{1/k}/\rho$ (ρ is the density) has a small variation. Such adiabatic flows, in the supersonic case, were dealt with earlier by Tarasov [4,5,6] with the additional assumption that $(1/v)\ (\partial^2 v/\partial p^2) =$ const, and without bringing out their relationship with potential flows. We point out, moreover, that the nonlinear problems of the type discussed here, and their general reduction to linear cases is studied in [7,8].

Here we give a proof to confirm the work in [1] for an example of adiabatic flow. Further on, a generalised approximate method is discussed which can be applied to potential motions in supersonic vortex flow. To do this we construct an approximate general solution similar to that of Khristianovich [9], but containing three and not two arbitrary functions. A solution is given for the certain boundary-value problems without shock and also for several flow problems with shock.

1. Statement of the problem. Let us take the plane vortex gas-flow equation, deduced by Sedov [2] in Rudnev's form [3]:

$$v\frac{\partial^2\psi}{\partial p^2}-\frac{\partial^2 v}{\partial p^2}\frac{\partial^2\psi}{\partial\theta^2}+2\frac{\partial v}{\partial p}\frac{\partial\psi}{\partial p}+\frac{\partial v}{\partial\psi}\left(\frac{\partial\psi}{\partial p}\right)^2-\frac{\partial^3 v}{\partial p^2\partial\psi}\left(\frac{\partial\psi}{\partial\theta}\right)^2=0 \qquad (1.1)$$

$$dz=d(x+iy)=-\rho_0 e^{i\theta}\left[\left(v\frac{\partial\psi}{\partial p}+i\frac{\partial v}{\partial p}\frac{\partial\psi}{\partial\theta}\right)d\theta+\left(\frac{\partial^2 v}{\partial p^2}\frac{\partial\psi}{\partial\theta}+i\frac{\partial v}{\partial p}\frac{\partial\psi}{\partial p}\right)dp\right] \qquad (1.2)$$

Here θ is the angle the velocity vector makes with the x axis, ρ_0 is some constant with the dimensions of density; the velocity v is determined from the Bernouilli integral

$$\frac{v^2}{2} + \frac{k}{k-1} p^{\frac{k-1}{k}} \vartheta(\psi) = i_0 \qquad (1.3)$$

where the total heat of stagnation i_0 = const.

Introduce the function $\psi = \psi/a_*$ and the dimensionless quantities

$$\bar{v} = \frac{v}{a_*}, \qquad \bar{\rho} = \frac{\rho}{\rho_0}, \qquad \bar{p} = \frac{p}{\rho_0 a_*^2}, \qquad \overline{\vartheta(\psi)} = (\rho_0 a_*^2)^{\frac{k-1}{k}} \frac{\vartheta(\psi)}{i_0} \qquad (1.4)$$

where a_* is the critical velocity of sound. For simplicity, the dashes will be omitted in what follows. Then, when we substitute the quantities deduced in this manner into (1.1) and (1.2), the latter retain their form except that ρ_0 in (1.2) disappears. Formula (1.3) and the adiabatic equations will be written thus:

$$v(p, \psi) = \left[\frac{k+1}{k-1}\left(1 - \frac{k}{k-1} p^{\frac{k-1}{k}} \vartheta(\psi)\right)\right]^{\frac{1}{2}}, \qquad \vartheta(\psi) = 2\frac{k-1}{k+1} p^{\frac{1}{k}} / \rho \qquad (1.5)$$

It is assumed, as before in [4], that the function $\vartheta(\psi)$ can be represented as

$$\vartheta(\psi) = \vartheta_0 (1 + \vartheta_1(\psi)) \qquad (1.6)$$

where ϑ_0 is its main part which is constant, whilst $\vartheta_1(\psi)$ is a small variable quantity whose square can be neglected in comparison with unity. If we take ρ_0 to be the density at the stagnation point on the streamline $\psi = \psi_0$, where $\vartheta_1(\psi_0) = 0$, the value of ϑ_0 will be determined from (1.5) with the condition that at this point $\rho = 1$, $p = (k+1)/2k$. For air $k = 1.4$ and $\vartheta_0 = 0.2986$.

Let us expand the function $v(p, \psi)$ (1.5) into a series in powers of the small parameter $\vartheta_1(\psi)$ and limit it to two terms only. Then we have

$$v(p, \psi) = \lambda_0(p) + \lambda_1(p) \vartheta_1(\psi) \qquad (1.7)$$

where

$$\lambda_0(p) = \left[\frac{k+1}{k-1}\left(1 - \frac{k}{k-1} p^{\frac{k-1}{k}} \vartheta_0\right)\right]^{\frac{1}{2}}$$

$$\lambda_1(p) = \frac{\lambda_0(p)}{2}\left[1 - \frac{k+1}{k-1} \frac{1}{\lambda_0^2(p)}\right] \qquad (1.8)$$

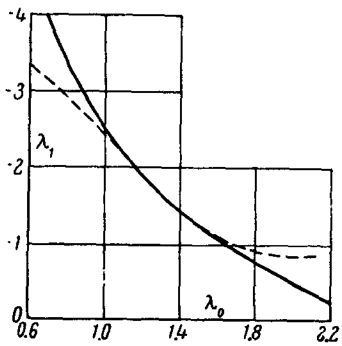

FIG. 1.

Furthermore, let us assume that the quantity $(1/v)(\partial^2 v/\partial p^2)$ is independent of ψ. For this we should have [2]

$$\frac{\lambda_0''(p)}{\lambda_0(p)} = \frac{\lambda_1''(p)}{\lambda_1(p)} = \frac{1}{v}\frac{\partial^2 v}{\partial p^2} \tag{1.9}$$

Here the strokes denote the operation of differentiating. From (1.9) it follows that

$$\lambda_1(p) = \lambda_0(p)\left[m_1 + m\int\frac{dp}{[\lambda_0(p)]^2}\right] \tag{1.10}$$

where m and m_1 are integration constants, whilst the integral, with k = 1.4 is

$$\int\frac{dp}{[\lambda_0(p)]^2} = -\left(\frac{t^5}{5} + \frac{t^3}{3} + t + \frac{1}{2}\ln\frac{1-t}{1+t}\right), \qquad t = 1\left(-\frac{k-1}{k+1}\lambda_0^2(p)\right)^{\frac{1}{2}}$$

Making use of our choice of constants m and m_1 we replace function $\lambda_1(p)$ (1.8) approximately by function (1.10) so that there are two common points, or a common tangent at one point; for $0.7 < \lambda_0 < 2.2$ this gives good approximation over large ranges of λ_0. For $\lambda_0 = 1.309$, if $m = -6.241$ and $m_1 = -0.3868$, we get by this means a second-order tangency. For this case, on Fig. 1, the full line shows the accurate relation $\lambda_1(\lambda_0)$ (1.8) and the broken line represents the approximation from Formula (1.10).

2. Reduction of the equation of vortex motion to a potential-flow equation. We introduce a new function ψ^* by formula

$$\lambda_0(p)\psi^* = \int_0^\psi v(p, \psi)d\psi \tag{2.1}$$

Differentiating this equation with respect to p and θ we get

$$\lambda_0\frac{\partial\psi^*}{\partial p} + \lambda_0'\psi^* = v\frac{\partial\psi}{\partial p} + \int_0^\psi\frac{\partial v}{\partial p}d\psi \tag{2.2}$$

$$\lambda_0\frac{\partial\psi^*}{\partial\theta} = v\frac{\partial\psi}{\partial\theta} \tag{2.3}$$

$$\lambda_0\frac{\partial^2\psi^*}{\partial p^2} + 2\lambda_0'\frac{\partial\psi^*}{\partial p} = v\frac{\partial^2\psi}{\partial p^2} + 2\frac{\partial v}{\partial p}\frac{\partial\psi}{\partial p} + \frac{\partial v}{\partial\psi}\left(\frac{\partial\psi}{\partial p}\right)^2 \tag{2.4}$$

$$\lambda_0\frac{\partial^2\psi^*}{\partial\theta^2} = v\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial v}{\partial\psi}\left(\frac{\partial\psi}{\partial\theta}\right)^2 \tag{2.5}$$

On subtracting Equation (2.5), multiplied term by term by the limit relations (1.9), from (2.4), we arrive at an expression, on the R.H.S. of which there will be the operator (1.1). Then the L.H.S. gives us a linear equation for the function ψ^*

$$\lambda_0(p)\frac{\partial^2\psi^*}{\partial p^2}-\lambda_0''(p)\frac{\partial^2\psi^*}{\partial\theta^2}+2\lambda_0'(p)\frac{\partial\psi^*}{\partial p}=0 \qquad (2.6)$$

which replaces the quasi-linear equation (1.1). Obviously (2.6) coincides with a potential-flow equation, for which $\lambda_0(p)$ is the velocity and ψ^* is the stream function.

To transform the formulas (1.2) we introduce the function

$$U(\psi)=\int_0^\psi\left(\lambda_0\frac{\partial v}{\partial p}-\lambda_0'v\right)d\psi=m\int_0^\psi\vartheta_1(\psi)\,d\psi=m\chi(\psi) \qquad (2.7)$$

It follows from (2.2) and (2.7) that

$$\lambda_0\frac{\partial\psi^*}{\partial p}-\frac{1}{\lambda_0}U(\psi)=v\frac{\partial\psi}{\partial p} \qquad (2.8)$$

Into (1.2) we substitute the derivatives of v with respect to p from (2.7) and (1.9) and then using (2.8) and (2.3) we eliminate the products $v(\partial\psi/\partial p)$ and $v(\partial\psi/\partial\theta)$.

Then (1.2) reduces to the expression

$$dz=dz^*+dz_1 \qquad (2.9)$$

where dz^* is the formula for potential flow

$$dz^*=-e^{i\theta}\left[\left(\lambda_0\frac{\partial\psi^*}{\partial p}+i\lambda_0'\frac{\partial\psi^*}{\partial\theta}\right)d\theta+\left(\lambda_0''\frac{\partial\psi^*}{\partial\theta}+i\lambda_0'\frac{\partial\psi^*}{\partial p}\right)dp\right] \qquad (2.10)$$

whilst

$$z_1=-ie^{i\theta}\lambda_0^{-1}(p)\,m\chi(\psi) \qquad (2.11)$$

Thus, the integration of the vortex motion equation is reduced to an analogous potential-flow problem.

In accordance with the properties of implicit functions, the relation between the stream-functions ψ and ψ^* (2.1) will be unique. Furthermore, along the line p = const these functions either grow or decrease simultaneously, inasmuch as from (2.1) $\partial\psi^*/\partial\psi > 0$. It follows from (2.1) and (2.9) that the streamlines of the vortex motion at which $\chi(\psi) = 0$ ($z_1 = 0$) coincide with the corresponding lines for potential flow. In particular, the lines $\psi = 0$ and $\psi^* = 0$ coincide.

In this manner, to each vortex motion of a gas there can be "matched" a single corresponding potential flow and vice versa. Thus, the solution of the boundary-value problem without shock for vortex motion can be replaced by a solution in the same region p, θ of a potential problem whose boundary conditions are determined from those given for ψ from (2.1).

Having found the potential-flow solution, on reversing the process back to (2.1) we obtain the solution of the vortex problem. It is evident that the theorems of the existence and the uniqueness of the solution, when proved for potential flows, also apply to the given vortex flows.

3. Approximate integration of the equations of vortex supersonic gas flow. Now, dealing only with supersonic motion, we go over in Equation (2.6) to the characteristic variables

$$2\xi = \sigma - \theta, \quad 2\eta = \sigma + \theta, \qquad \sigma = -\int_{p_*}^{p} \mu(p)\,dp, \qquad \mu = \sqrt{\frac{\lambda_0''(p)}{\lambda_0(p)}} \tag{3.1}$$

where the critical pressure p_* is, evidently, identical for vortex and for potential motion for any value of ψ, because it is determined from the condition that (1.9) vanishes [2].

Equation (2.6) appears in the well-known form [9]

$$\frac{\partial^2\psi^*}{\partial\xi\,\partial\eta} + \frac{1}{2}\frac{d\ln\sqrt{K_1}}{d\sigma}\left(\frac{\partial\psi^*}{\partial\xi} + \frac{\partial\psi^*}{\partial\eta}\right) = 0 \qquad \left(\sqrt{K_1} = \sqrt{-K} = \lambda_0^2\mu\right) \tag{3.2}$$

In this expression K is Chaplygin's function. This formula, together with (3.1), ties up $\sqrt{K_1}$, λ_0, μ, p as functions of σ. For adiabatic flow, according to (1.8), taking account of the values of ϑ_0, we have the familiar formulas

$$\sigma = \int_1^{\lambda_0} \sqrt{\frac{\lambda_0^2 - 1}{1 - \lambda_0^2/\nu}}\,\frac{d\lambda_0}{\lambda_0}, \qquad \sqrt{K_1} = \sqrt{\frac{\lambda_0^2 - 1}{(1 - \lambda_0^2/\nu)^\nu}} \qquad \left(\nu = \frac{k+1}{k-1}\right) \tag{3.3}$$

The following approximation was suggested by Khristianovich (3.4)

$$\sqrt{K_1} = a_0^2(\sigma + c_0)^2, \quad \lambda_0 = \frac{a_0(\sigma + c_0)}{N\sin(\sigma + \varepsilon)}, \quad \mu = \frac{\sqrt{K_1}}{\lambda_0^2}, \quad p = \frac{1}{N^2}(\cot(\sigma+\varepsilon)+n)$$

which, with the constants $a_0^2 = 18.5$, $c_0 = 0.185$, $N = 2.398$, $\epsilon = 0.3347$, $n = -0.3161$, gives a good approximation in the neighborhood of the point $\sigma = 0.197$ to 0.20.

The general solution for this case is as follows:

$$(\xi + \eta + c_0)\psi^* = \Phi_1(\xi) + \Phi_2(\eta) \tag{3.5}$$

where $\Phi_1(\xi)$ and $\Phi_2(\eta)$ are arbitrary functions.

Now let us study the vortex motion which corresponds to a given potential motion. From (1.10), (3.1) and (3.4) we get

$$\lambda_1 = \lambda_0(\sigma)\left[m_1 - m\int \frac{d\sigma}{\sqrt{K_1}}\right] = \frac{a_1(\sigma + c_1)}{N \sin(\sigma + \varepsilon)} \qquad \left(a_1 = a_0 m_1, \quad c_1 = c_0 + \frac{m}{a_0^2 m_1}\right) \quad (3.6)$$

Taking $a_1 = 0.03868$, $c_1 = -48.74$ we approximate the function λ_1 (1.8) by the function (3.6) which is tangent to it at the point $\sigma = 0.197$. In Fig. 2 we give a comparison between the exact relations between λ_0 and λ_1 and σ (full lines) and the approximate ones from Formulas (3.4) and (3.6).

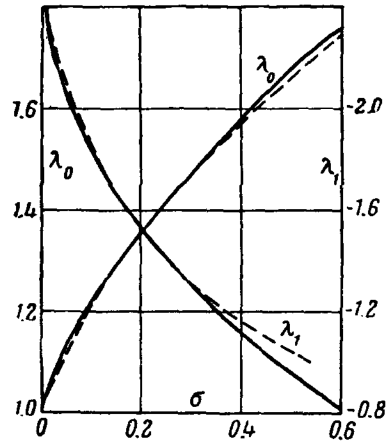

FIG. 2.

From (2.1) and (3.4)-(3.6) we obtain the approximate general solution to the equation of vortex motion

$$(\xi + \eta + c_0)\psi + a(\xi + \eta + c_1)\chi(\psi) = \\ = \Phi_1(\xi) + \Phi_2(\eta) \qquad (3.7)$$

where $a = a_1/a_0$ and $\chi(\psi)$ is determined in (2.7). It is clear from (2.7) that for any finite value of ψ, $\chi(\psi)$ is small, together with $\vartheta_1(\psi)$. Then, if we replace the argument of the small function $\chi(\psi)$ by the nearly equal quantity ψ^* (3.5), from (3.7) we obtain, if necessary, an implicit expression $\psi = \psi(\xi, \eta)$.

The formula for transformation to the physical plane was obtained by us in the form (2.9). Let us put dz^* (2.10) in characteristic coordinates

$$dz^* = \mu e^{i\theta}\left[\left(-\lambda_0 + i\frac{d\lambda_0}{d\sigma}\right)\frac{\partial\psi^*}{\partial\xi}d\xi + \left(\lambda_0 + i\frac{d\lambda_0}{d\sigma}\right)\frac{\partial\psi^*}{\partial\eta}d\eta\right] \qquad (3.8)$$

Inserting into this the approximate expressions μ, λ_0 and the partial derivatives of function ψ^* from (3.5), and working out the quadratures we obtain

$$z^* = ia_0\Big\{a_0 e^{i(\eta-\xi)}\lambda_0^{-1}(\xi + \eta)[\Phi_1(\xi) + \Phi_2(\eta)] - \\ - N\left[\int e^{-i(2\xi+\varepsilon)}\Phi_1'(\xi)\,d\xi + \int e^{i(2\eta+\varepsilon)}\Phi_2'(\eta)\,d\eta\right]\Big\} \qquad (3.9)$$

Furthermore

$$z_1 = -ie^{i(\eta-\xi)}\lambda_0^{-1}(\xi + \eta)\,m\chi(\psi), \qquad m = a_0 a_1(c_1 - c_0) \qquad (3.10)$$

and

$$z = z^* + z_1 \qquad (3.11)$$

In this way we have obtained an approximate general solution for a vortex supersonic flow in the (ξ, η) plane which depends on three arbitrary functions $\Phi_1(\xi)$, $\Phi_2(\xi)$ and $\chi(\psi)$, and also a formula for transferring to the physical plane. In an exactly similar way, approximate general solutions can be found which correspond to other approximations of Chaplygin functions, both for cases of subsonic and supersonic flow.

4. Boundary-value problems for flows without shock. It is possible to establish four basic boundary-value problems for such flows: Cauchy's problem, Goursat's problem, flow with a free surface and flow past a solid wall. The function $\chi(\psi)$, which represents the vortex distribution, should, in this case be given. When the general solution (3.7), is simple in form, all these problems can be solved, and, indeed, directly, without introducing potential flow as recommended in Section 2. The same actual difficulties will arise, because functions $\Phi_1(\xi)$ and $\Phi_2(\eta)$ are the same for both of the flows.

As an example we will examine the flow past a solid wall given by the equations $x = X(\theta)$ and $y = Y(\theta)$, if, on the characteristic $\xi = \xi_0 =$ const which intersects it, we are given $\psi = \psi_2(\eta)$ and $\chi(\psi)$.

As usual, we take the wall to be the line $\psi = 0$. Then, on it $\chi(\psi) = 0$ and

$$\Phi_1(\xi) + \Phi_2(\omega) = 0 \tag{4.1}$$

where $\eta = \omega(\xi)$ is the equation of the solid wall in the ξ, η plane. From (3.7), by the condition on the characteristic $\xi = \xi_0$ we have

$$\Phi_2(\eta) = (\xi_0 + \eta + c_0)\psi_2(\eta) + a(\xi_0 + \eta + c_1)\chi[\psi_2(\eta)] - \Phi_1(\xi_0) \tag{4.2}$$

The function $\Phi_1(\xi)$ is determined from (4.1) if $\omega(\xi)$ is known. A differential equation for finding $\omega(\xi)$ is easily obtained from (3.9) because $\chi(\psi) = 0$, from the condition of total differentiation (4.1) with respect to ξ by separating the real and imaginary parts. In symmetrical form this is

$$[X'(\omega - \xi)\cos(\omega - \xi) + Y'(\omega - \xi)\sin(\omega - \xi)][\omega'(\xi) - 1] - \\ - 2Na_0 \sin(\xi + \omega + \varepsilon)\Phi_2'(\omega)\omega'(\xi) = 0 \tag{4.3}$$

It is just as simple to solve all the other problems.

5. Conditions at the shock waves. To find the solution to flow problems involving shock waves let us examine the conditions which have to be fulfilled. These are, first of all, conditions of dynamic compatibility which for steady conditions are as follows:

$$-\rho_1 v_{1n}[v_n] = [p], \qquad [v_\tau] = 0, \qquad [\rho v_n] = 0, \qquad [i_0] = 0 \tag{5.1}$$

where the symbol [ ] denotes the discontinuity, index 1 relates to parameters before the shock, v_τ and v_n are velocity components tangential and normal to the shock wave drawn in the direction of the flow of the gas. If β is the angle between the shock wave and the x-axis, then $v_\tau = v\cos(\beta-\theta)$ and $v_n = v\sin(\beta-\theta)$.

Let us express the conditions (5.1) in dimensionless form. For ρ_0 we take the stagnation density at some streamline $\psi=\psi_0$ behind the wave front and we assume i_0 and a_* to remain the same before and after the shock. Through these the fourth condition will be fulfilled whilst the first three remain as before.

We will limit our study from now on to a steady oncoming flow along the direction of the x-axis, i.e. $\theta_1 = 0$. Denoting $\tan\beta$ by Γ from (5.1), after some simple transformation we obtain

$$\frac{\rho_1 v_1^2\,\Gamma\sin\theta}{\Gamma\sin\theta+\cos\theta}=[p],\quad v_1=v\,(\Gamma\sin\theta+\cos\theta),\quad \rho_1 v_1\Gamma=\rho v\,(\Gamma\cos\theta-\sin\theta)\quad (5.2)$$

Here we only examine those shock waves for which the flow remains supersonic on traversing them, and the function $\vartheta(\psi)$ which interests us hardly alters. Then, if we represent the velocity $v(p,\psi)$ in the form (1.7), from the first and the second condition we arrive at

$$\frac{1}{\Gamma}=\left(\frac{\rho_1 v_1^2}{[p]}-1\right)\tan\theta \quad (5.3)$$

$$\chi'(\psi)=\frac{1}{\lambda_1}\left\{\frac{v_1}{\cos\theta}\left(1-\frac{[p]}{\rho_1 v_1^2}\right)-\lambda_0\right\} \quad (5.4)$$

In Formulas (5.2)-(5.4) we will regard λ_0, λ_1, p as functions of σ (3.1). Then, in the third condition (5.2), in accordance with the Bernouilli integral $1/\rho = \mu v\,(\partial v/\partial\sigma)$, and eliminating Γ and $\chi'(\psi)$ we arrive at a quadratic in $\cos\theta$ which yields

$$\cos\theta=\frac{m\,[p]}{2v_1\lambda_1}+\left[\left(\frac{m\,[p]}{2v_1\lambda_1}\right)^2+\left(1-\frac{[p]}{\rho_1 v_1^2}\right)\left(\frac{\mu\,[p]}{\lambda_1}\,\frac{d\lambda_1}{d\sigma}+1\right)\right]^{1/2}=H(\sigma)\quad (5.5)$$

where λ_1 is expressed by Formula (1.10).

Only a plus sign is admissible in front of the square root, as this corresponds to supersonic flow behind the shock wave; when $[p]=0$, $\cos\theta = 1$, $\theta=\theta_1=0$. The expression (5.5) represents a relationship between σ and θ on the shock wave. Evidently, then, $\Gamma=\Gamma(\sigma)$ and $\chi'(\psi)=f(\sigma)$.

From (3.1) and (5.5) we have

$$2\xi=\sigma-\cos^{-1}H(\sigma),\quad 2\eta=\sigma+\cos^{-1}H(\sigma)\quad (5.6)$$

which may be regarded as parametric equations of a shock wave in the characteristic plane, elimination of σ from which gives $\eta = g(\xi)$ or $\xi = h(\eta)$. All the functions on the shock wave can be expressed through one of the parameters σ, ξ or η.

From Formula (1.2) with condition (1.9) after going over to characteristic variables, we obtain

$$\begin{aligned} dx &= \mu\left[-\left(v\cos\theta + \frac{\partial v}{\partial\sigma}\sin\theta\right)\frac{\partial\psi}{\partial\xi}d\xi + \left(v\cos\theta - \frac{\partial v}{\partial\sigma}\sin\theta\right)\frac{\partial\psi}{\partial\eta}d\eta\right] \\ dy &= \mu\left[\left(-v\sin\theta + \frac{\partial v}{\partial\sigma}\cos\theta\right)\frac{\partial\psi}{\partial\xi}d\xi + \left(v\sin\theta + \frac{\partial v}{\partial\sigma}\cos\theta\right)\frac{\partial\psi}{\partial\eta}d\eta\right] \end{aligned} \qquad (5.7)$$

We will consider here ξ and η to be connected by the expressions (5.6). Then (5.7) might be a parametric equation for the shock wave in the physical plane. As, on the wave, $dy/dx = \Gamma$, then from (5.7) and (5.2) it is easy to derive the equation

$$\left(\frac{\partial\psi}{\partial\xi}d\xi - \frac{\partial\psi}{\partial\eta}d\eta\right) + P(\xi,\eta)\left(\frac{\partial\psi}{\partial\xi}d\xi + \frac{\partial\psi}{\partial\eta}d\eta\right) = 0 \qquad \left(P(\xi,\eta) = \frac{v_1\sin\theta}{v\mu[p]}\right) \qquad (5.8)$$

To this we add one more obvious equation

$$\frac{\partial\psi}{\partial\xi}d\xi + \frac{\partial\psi}{\partial\eta}d\eta = d\psi \qquad (5.9)$$

On the shock wave $\psi = \psi^\circ(\sigma) = \psi_1^\circ(\xi) = \psi_2^\circ(\eta)$, $\chi'(\psi) = f(\sigma) = f_1(\xi)$.

When solving boundary-value problems with the help of the general solution (3.7) it is natural to take the functions entering (5.3) to (5.8) in the approximate form given by (3.4) and (3.6). This gives satisfactory results until $[p] = p(\sigma) - p_1$ is so small that the approximate difference expression gives a significant error (about 10%), although the error in determining (σ) itself may not be great. If necessary this difficulty might be overcome by constructing an approximation at the point σ corresponding to $p = p_1$.

6. Boundary-value problems for flows with shocks. In such problems the function $\chi(\psi)$ is not known beforehand in the approximate general solution (3.7) and it has to be found together with $\Phi_1(\xi)$ and $\Phi_2(\eta)$ from the boundary conditions and the conditions on the shock wave.

Let us examine two problems where we determine the motion of gas in the region bounded by the shock wave and two intersecting characteristics of opposing families. For simplicity we assume that on the streamline passing through O, the intersection of the shock wave with one of the characteristics is $\psi = \psi_0 = 0$. The parameters of the steady stream in front of the wave are known.

Problem 1. Assume that the shape of the shock wave is defined by the equations $x = x_\beta(\Gamma)$ and $y = y_\beta(\Gamma)$.

To solve the problem one must know the values of the dimensionless parameters p_1 and ρ_1, which depend on the stagnation density ρ_0 behind the shock on the streamline $\psi_0 = 0$. We find this from the condition $[i_0] = 0$, i.e. $\rho_0/\rho_{01} = p_0/p_{01}$, because for β_0 known at point O the R.H.S. of the equation can be worked out exactly [3]. In particular, if the shock wave is tangent to the characteristic at point O, then $\rho_0 = \rho_{01}$. We find the values of σ_0 and θ_0 at point O from (5.3) and (5.5) for $\Gamma = \Gamma_0$.

In view of the fact that $\Gamma = \Gamma(\sigma)$ the shock wave equation can be written $x = x^0(\sigma)$ and $y = y^0(\sigma)$. Additionally, on it we have evidently $\psi = \rho_1 v_1(y^0 - y_0)$ where y_0 is the coordinate of point O. If we introduce (5.4) and integrate it we obtain

$$\psi = \psi^\circ(\sigma) = \rho_1 v_1(y^\circ - y_0); \qquad \chi = \int_{\sigma_0}^{\sigma} f(\sigma)\,\psi^{\circ\prime}(\sigma)\,d\sigma \tag{6.1}$$

These equations give a parametric representation of the function $\chi(\psi)$ along the shock wave. This is sufficient for finding the solution in the region occupied by streamlines which intersect the wave, as, for instance, in the problem of finding the shape of a profile. To find the solution for the other part of the region we should be given the function $\chi(\psi)$ there.

From (5.8) and (5.9) we have

$$\frac{\partial\psi}{\partial\xi} = \frac{1}{2}(1 - P(\xi,\eta))\,\psi_1^{\circ\prime}(\xi), \qquad \frac{\partial\psi}{\partial\eta} = \frac{1}{2}(1 + P(\xi,\eta))\,\psi_2^{\circ\prime}(\eta) \tag{6.2}$$

Thus, ψ with its partial derivatives and $\chi(\psi)$ are found on the shock wave and the problem reduces to a Cauchy problem.

It follows from the general solution of (3.7) that

$$\begin{aligned} \Phi_1'(\xi) &= \psi + a\chi(\psi) + [(\xi + \eta + c_0) + a(\xi + \eta + c_1)\chi'(\psi)]\frac{\partial\psi}{\partial\xi} \\ \Phi_2'(\eta) &= \psi + a\chi(\psi) + [(\xi + \eta + c_0) + a(\xi + \eta + c_1)\chi'(\psi)]\frac{\partial\psi}{\partial\eta} \end{aligned} \tag{6.3}$$

If we express the R.H.S. of the first formula by ξ and of the second by η and work out the quadratures, we find the functions $\Phi_1(\xi)$ and $\Phi_2(\eta)$. The integration constants are found, as usual, from the conditions at point O.

Problem 2. Suppose that the shape of the shock wave is not known, but we are given the flow parameters on one of the characteristics (for instance $\eta = \eta_0$) which does not intersect the streamlines which go through a section of the wave, i.e. $\psi = \psi_1(\xi)$ and $\chi(\psi)$ are given.

From the general solution (3.7) for $\eta = \eta_0$ we find

$$\Phi_1(\xi) = (\xi + \eta_0 + c_0)\psi_1(\xi) + a(\xi + \eta_0 + c_1)\chi(\psi_1) - \Phi_2(\eta_0) \tag{6.4}$$

Eliminating $\partial\psi/\partial\eta$ from (5.8) and (5.9) and replacing $\partial\psi/\partial\xi$ from (6.3) we arrive at the equation

$$Q(\xi)\frac{d\psi_1^\circ}{d\xi} + \psi_1^\circ + a\chi(\psi) - \Phi_1'(\xi) = 0 \tag{6.5}$$

where

$$Q(\xi) = \frac{\xi + g(\xi) + c_0}{2}\frac{v}{\lambda_0}[1 - P(\xi, g(\xi))]$$

Adding to this (5.4) in this form

$$\frac{d\chi}{d\xi} = f_1(\xi)\frac{d\psi_1^\circ}{d\xi} \tag{6.6}$$

we obtain a system of ordinary linear equations for finding $\psi_1^0(\xi)$ and χ. Its solution is

$$\psi = \psi_1^\circ(\xi) = \int_{\xi_0}^{\xi}\frac{R(\xi)}{Q(\xi)}d\xi, \qquad \chi = \int_{\xi_0}^{\xi}\frac{f_1(\xi)R(\xi)}{Q(\xi)}d\xi \tag{6.7}$$

where ξ_0 is the value of ξ at point O, and

$$R(\xi) = \Phi_1'(\xi_0)\exp\left(-\int_{\xi_0}^{\xi}\frac{af_1+1}{Q}d\xi\right) - \\ - \exp\left(\int_{\xi_0}^{\xi}\frac{af_1+1}{Q}d\xi\right)\int_{\xi_0}^{\xi}\Phi_1''(\xi)\exp\left(\int_{\xi_0}^{\xi}\frac{af_1+1}{Q}d\xi\right)d\xi$$

The function $\chi(\psi)$ is therefore determined parametrically. Then, from (3.7) taken on the shock wave we find

$$\Phi_2(\eta) = (h(\eta) + \eta + c_0)\psi_2^\circ(\eta) + a(h(\eta) + \eta + c_1)\chi(\psi_2^\circ) - \Phi_1[h(\eta)] \tag{6.8}$$

When $\Phi_1(\xi)$ and $\Phi_2(\eta)$ are substituted in the general solution the constant $\Phi_2(\eta_0)$ vanishes. In this manner the solution of both our problems reduces to calculating quadratures.

Knowing the solutions to the vortex problems without shock and the problems just discussed involving shock, we are able to study the more important supersonic gas flows with entropy functions which only change slightly, including flow round profiles etc.

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